e. This integral cannot be evaluated without knowing the intervals on which f is positive and negative. It could have any value greater than or equal to 10.

Related Exercises 39–44

QUICK CHECK 6 Evaluate $\int_{-1}^{2} x\,dx$ and $\int_{-1}^{2} |x|\,dx$ using geometry.

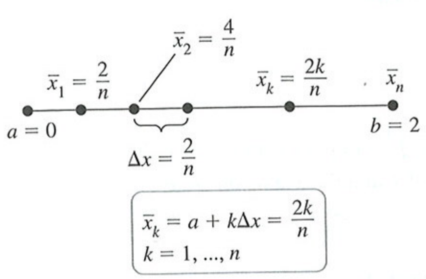
Evaluating Definite Integrals Using Limits

In Example 3 we used area formulas for trapezoids, triangles, and circles to evaluate definite integrals. Regions bounded by more general functions have curved boundaries for which conventional geometrical methods do not work. At the moment the only way to handle such integrals is to appeal to the definition of the definite integral and the summation formulas given in Theorem 5.1.

We know that if f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k)\Delta x_k$ for any partition of $[a, b]$ and any points $\bar{x}_k$. To simplify these calculations, we use equally spaced grid points and right Riemann sums. That is, for each value of n we let $\Delta x_k = \Delta x = \dfrac{b-a}{n}$ and $\bar{x}_k = a + k\,\Delta x$, for $k = 1, 2, \ldots, n$. Then, as $n \to \infty$ and $\Delta \to 0$,

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k)\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

EXAMPLE 6 Evaluating definite integrals Find the value of $\int_0^2 (x^3 + 1)\,dx$ by evaluating a right Riemann sum and letting $n \to \infty$.



SOLUTION Based on approximations found in Example 5, Section 5.1, we conjectured that the value of this integral is 6. To verify this conjecture, we now evaluate the integral exactly. The interval $[a, b] = [0, 2]$ is divided into n subintervals of length $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$, which produces the grid points

$$\bar{x}_k = a + k\,\Delta x = 0 + k \cdot \frac{2}{n} = \frac{2k}{n}, \qquad \text{for } k = 1, 2, \ldots, n.$$

Letting $f(x) = x^3 + 1$, the right Riemann sum is

$$\begin{aligned}
\sum_{k=1}^{n} f(\bar{x}_k)\Delta x &= \sum_{k=1}^{n}\left[\left(\frac{2k}{n}\right)^3 + 1\right]\frac{2}{n} \\
&= \frac{2}{n}\sum_{k=1}^{n}\left(\frac{8k^3}{n^3} + 1\right) && \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \\
&= \frac{2}{n}\left(\frac{8}{n^3}\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1\right) && \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \\
&= \frac{2}{n}\left[\frac{8}{n^3}\left(\frac{n^2(n+1)^2}{4}\right) + n\right] && \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \text{ and } \sum_{k=1}^{n} 1 = n;\ \text{Theorem 5.1} \\
&= \frac{4(n^2 + 2n + 1)}{n^2} + 2. && \text{Simplify.}
\end{aligned}$$

➤ An analogous calculation could be done using left Riemann sums or midpoint Riemann sums.

Now we evaluate $\int_0^2 (x^3 + 1)\,dx$ by letting $n \to \infty$ in the Riemann sum:

$$\begin{aligned}
\int_0^2 (x^3 + 1)\,dx &= \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{x}_k)\Delta x \\
&= \lim_{n \to \infty}\left[\frac{4(n^2 + 2n + 1)}{n^2} + 2\right] \\
&= 4\lim_{n \to \infty}\left(\frac{n^2 + 2n + 1}{n^2}\right) + \lim_{n \to \infty} 2 \\
&= 4(1) + 2 = 6
\end{aligned}$$

Therefore, $\int_0^2 (x^3 + 1)\,dx = 6$, confirming our conjecture in Example 5, Section 5.1.

Related Exercises 45–50

The Riemann sum calculations in Example 6 are tedious even if f is a simple function. For polynomials of degree 4 and higher, the calculations are much more challenging, and for rational and transcendental functions, advanced mathematical results are needed. The next section introduces more efficient methods for evaluating definite integrals.

SECTION 5.2 EXERCISES

Review Questions

1. Explain what net area means.
2. How do you interpret geometrically the definite integral of a function that changes sign on the interval of integration?
3. When does the net area of a region equal the area of a region? When does the net area of a region differ from the area of a region?
4. Suppose that $f(x) < 0$ on the interval $[a, b]$. Using Riemann sums, explain why the definite integral $\int_a^b f(x)\,dx$ is negative.
5. Use graphs to evaluate $\int_0^{2\pi} \sin x\,dx$ and $\int_0^{2\pi} \cos x\,dx$.
6. Explain how the notation for Riemann sums, $\sum_{k=1}^{n} f(\bar{x}_k)\Delta x$, corresponds to the notation for the definite integral, $\int_a^b f(x)\,dx$.
7. Give a geometrical explanation of why $\int_a^a f(x)\,dx = 0$.
8. Use Table 5.3 to rewrite $\int_1^6 (2x^3 - 4x)\,dx$ as the sum of two integrals.
9. Use geometry to find a formula for $\int_0^a x\,dx$, in terms of a.
10. If f is continuous on $[a, b]$ and $\int_a^b |f(x)|\,dx = 0$, what can you conclude about f?

Basic Skills

11–14. Approximating net area *The following functions are negative on the given interval.*

a. Sketch the function on the given interval.

b. Approximate the net area bounded by the graph of f and the x-axis on the interval using a left, right, and midpoint Riemann sum with $n = 4$.

11. $f(x) = -2x - 1;\ [0, 4]$
12. $f(x) = -4 - x^3;\ [3, 7]$
13. $f(x) = \sin 2x;\ [\pi/2, \pi]$
14. $f(x) = x^3 - 1;\ [-2, 0]$

15–18. Approximating net area *The following functions are positive and negative on the given interval.*

a. Sketch the function on the given interval.

b. Approximate the net area bounded by the graph of f and the x-axis on the interval using a left, right, and midpoint Riemann sum with $n = 4$.

c. Use the sketch in part (a) to show which intervals of $[a, b]$ make positive and negative contributions to the net area.

15. $f(x) = 4 - 2x;\ [0, 4]$
16. $f(x) = 8 - 2x^2;\ [0, 4]$
17. $f(x) = \sin 2x;\ [0, 3\pi/4]$
18. $f(x) = x^3;\ [-1, 2]$

19–22. Identifying definite integrals as limits of sums *Consider the following limits of Riemann sums of a function f on $[a, b]$. Identify f and express the limit as a definite integral.*

19. $\lim_{\Delta \to 0} \sum_{k=1}^{n} (\bar{x}_k^2 + 1)\Delta x_k;\ [0, 2]$
20. $\lim_{\Delta \to 0} \sum_{k=1}^{n} (4 - \bar{x}_k^2)\Delta x_k;\ [-2, 2]$
21. $\lim_{\Delta \to 0} \sum_{k=1}^{n} \bar{x}_k \ln \bar{x}_k \Delta x_k;\ [1, 2]$
22. $\lim_{\Delta \to 0} \sum_{k=1}^{n} |\bar{x}_k^2 - 1|\Delta x_k;\ [-2, 2]$

23–30. Net area and definite integrals *Use geometry (not Riemann sums) to evaluate the following definite integrals. Sketch a graph of the integrand, show the region in question, and interpret your result.*

23. $\int_0^4 (8 - 2x)\,dx$
24. $\int_{-4}^{2} (2x + 4)\,dx$

25. $\int_{-1}^{2} (-|x|)\,dx$

26. $\int_{0}^{2} (1 - |x|)\,dx$

27. $\int_{0}^{4} \sqrt{16 - x^2}\,dx$

28. $\int_{-1}^{3} \sqrt{4 - (x-1)^2}\,dx$

29. $\int_{0}^{4} f(x)\,dx$ where $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ 3x - 1 & \text{if } x > 2 \end{cases}$

30. $\int_{1}^{10} g(x)\,dx$ where $g(x) = \begin{cases} 4x & \text{if } 0 \le x \le 2 \\ -8x + 16 & \text{if } 2 < x \le 3 \\ -8 & \text{if } x > 3 \end{cases}$

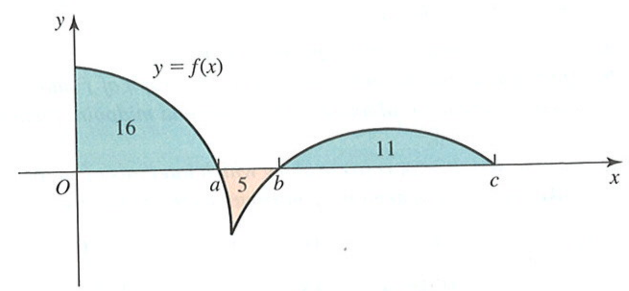
31–34. Net area from graphs *The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the following integrals.*

31. $\int_{0}^{a} f(x)\,dx$

32. $\int_{0}^{b} f(x)\,dx$

33. $\int_{a}^{c} f(x)\,dx$

34. $\int_{0}^{c} f(x)\,dx$



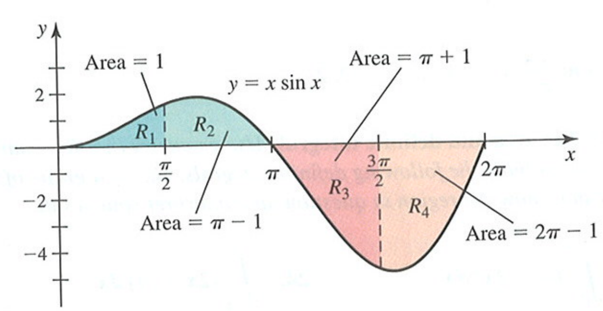
35–38. Net area from graphs *The accompanying figure shows four regions bounded by the graph of $y = x \sin x$: R_1, R_2, R_3, and R_4, whose areas are $1, \pi - 1, \pi + 1$, and $2\pi - 1$, respectively. (We verify these results later in the text.) Use this information to evaluate the following integrals.*

35. $\int_{0}^{\pi} x \sin x\,dx$

36. $\int_{0}^{3\pi/2} x \sin x\,dx$

37. $\int_{0}^{2\pi} x \sin x\,dx$

38. $\int_{\pi/2}^{2\pi} x \sin x\,dx$



39. **Properties of integrals** Use only the fact that $\int_0^4 3x(4-x)\,dx = 32$ and the definitions and properties of integrals to evaluate the following integrals, if possible.

a. $\int_{4}^{0} 3x(4-x)\,dx$

b. $\int_{0}^{4} x(x-4)\,dx$

c. $\int_{4}^{0} 6x(4-x)\,dx$

d. $\int_{0}^{8} 3x(4-x)\,dx$

40. **Properties of integrals** Suppose $\int_1^4 f(x)\,dx = 8$ and $\int_1^6 f(x)\,dx = 5$. Evaluate the following integrals.

a. $\int_{1}^{4} (-3f(x))\,dx$

b. $\int_{1}^{4} 3f(x)\,dx$

c. $\int_{6}^{4} 12f(x)\,dx$

d. $\int_{4}^{6} 3f(x)\,dx$

41. **Properties of integrals** Suppose $\int_0^3 f(x)\,dx = 2$, $\int_3^6 f(x)\,dx = -5$, and $\int_3^6 g(x)\,dx = 1$. Evaluate the following integrals.

a. $\int_{0}^{3} 5f(x)\,dx$

b. $\int_{3}^{6} [-3g(x)]\,dx$

c. $\int_{3}^{6} (3f(x) - g(x))\,dx$

d. $\int_{6}^{3} [f(x) + 2g(x)]\,dx$

42. **Properties of integrals** Suppose that $f(x) \ge 0$ on $[0, 2]$, $f(x) \le 0$ on $[2, 5]$, $\int_0^2 f(x)\,dx = 6$, and $\int_2^5 f(x)\,dx = -8$. Evaluate the following integrals.

a. $\int_{0}^{5} f(x)\,dx$

b. $\int_{0}^{5} |f(x)|\,dx$

c. $\int_{2}^{5} 4|f(x)|\,dx$

d. $\int_{0}^{5} (f(x) + |f(x)|)\,dx$

43–44. Using properties of integrals *Use the value of the first integral I to evaluate the two given integrals.*

43. $I = \int_{0}^{1} (x^3 - 2x)\,dx = -\frac{3}{4}$

a. $\int_{0}^{1} (4x - 2x^3)\,dx$

b. $\int_{1}^{0} (2x - x^3)\,dx$

44. $I = \int_{0}^{\pi/2} (\cos\theta - 2\sin\theta)\,d\theta = -1$

a. $\int_{0}^{\pi/2} (2\sin\theta - \cos\theta)\,d\theta$

b. $\int_{\pi/2}^{0} (4\cos\theta - 8\sin\theta)\,d\theta$

45–50. Limits of sums *Use the definition of the definite integral to evaluate the following definite integrals. Use right Riemann sums and Theorem 5.1.*

45. $\int_{0}^{2} (2x + 1)\,dx$

46. $\int_{1}^{5} (1 - x)\,dx$

47. $\int_{3}^{7} (4x + 6)\,dx$

48. $\int_{0}^{2} (x^2 - 1)\,dx$

49. $\int_{1}^{4} (x^2 - 1)\,dx$

50. $\int_{0}^{2} 4x^3\,dx$

Further Explorations

51. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If f is a constant function on the interval $[a, b]$, then the right and left Riemann sums give the exact value of $\int_a^b f(x)\,dx$ for any n.

b. If f is a linear function on the interval $[a, b]$, then a midpoint Riemann sum gives the exact value of $\int_a^b f(x)\,dx$ for any n.

c. $\int_0^{2\pi/a} \sin ax\,dx = \int_0^{2\pi/a} \cos ax\,dx = 0$ (*Hint:* Graph the functions and use properties of trigonometric functions).

d. If $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$, then f is a constant function.

e. Property 4 of Table 5.3 implies that $\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$.

52–55. Approximating definite integrals *Complete the following steps for the given integral and the given value of n.*

a. Sketch the graph of the integrand on the interval of integration.

b. Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$, assuming a regular partition.

c. Calculate the left and right Riemann sums for the given value of n.

d. Determine which Riemann sum (left or right) underestimates the value of the definite integral and which overestimates the value of the definite integral.

52. $\int_{0}^{2} (x^2 - 2)\,dx;\ n = 4$

53. $\int_{3}^{6} (1 - 2x)\,dx;\ n = 6$

54. $\int_{0}^{\pi/2} \cos x\,dx;\ n = 4$

55. $\int_{1}^{7} \frac{1}{x}\,dx;\ n = 6$

56–60. Approximating definite integrals with a calculator *Consider the following definite integrals.*

a. Write the left and right Riemann sums in sigma notation for n = 20, 50, and 100. Then evaluate the sums using a calculator.

b. Based upon your answers to part (a), make a conjecture about the value of the definite integral.

56. $\int_{4}^{9} 3\sqrt{x}\,dx$

57. $\int_{0}^{1} (x^2 + 1)\,dx$

58. $\int_{0}^{1} \tan\left(\frac{\pi x}{4}\right)dx$

59. $\int_{0}^{1} e^x\,dx$

60. $\int_{-1}^{1} \cos\left(\frac{x\pi}{2}\right)dx$

61–64. Riemann sums with midpoints and a calculator *Consider the following definite integrals.*

a. Write the midpoint Riemann sum in sigma notation for an arbitrary value of n.

b. Evaluate each sum using a calculator with n = 20, 50, and 100. Use these values to estimate the value of the integral.

61. $\int_{1}^{4} 2\sqrt{x}\,dx$

62. $\int_{-1}^{2} \sin\left(\frac{x\pi}{4}\right)dx$

63. $\int_{0}^{4} (4x - x^2)\,dx$

64. $\int_{0}^{1/2} \sin^{-1} x\,dx$

65. **More properties of integrals** Consider two functions f and g on $[1, 6]$ such that $\int_1^6 f(x)\,dx = 10$, $\int_1^6 g(x)\,dx = 5$, $\int_4^6 f(x)\,dx = 5$, and $\int_1^4 g(x)\,dx = 2$. Evaluate the following integrals.

a. $\int_{1}^{4} 3f(x)\,dx$

b. $\int_{1}^{6} (f(x) - g(x))\,dx$

c. $\int_{1}^{4} (f(x) - g(x))\,dx$

d. $\int_{4}^{6} (g(x) - f(x))\,dx$

e. $\int_{4}^{6} 8g(x)\,dx$

f. $\int_{4}^{1} 2f(x)\,dx$

66–69. Area versus net area *Graph the following functions. Then use geometry (not Riemann sums) to find the area and the net area of the region described.*

66. The region between the graph of $y = 4x - 8$ and the x-axis for $-4 \le x \le 8$

67. The region between the graph of $y = -3x$ and the x-axis for $-2 \le x \le 2$

68. The region between the graph of $y = 3x - 6$ and the x-axis for $0 \le x \le 6$

69. The region between the graph of $y = 1 - |x|$ and the x-axis for $-2 \le x \le 2$

70–73. Area by geometry *Use geometry to evaluate the following integrals.*

70. $\int_{-2}^{3} |x + 1|\,dx$

71. $\int_{1}^{6} |2x - 4|\,dx$

72. $\int_{1}^{6} (3x - 6)\,dx$

73. $\int_{-6}^{4} \sqrt{24 - 2x - x^2}\,dx$

Additional Exercises

74. **Integrating piecewise continuous functions** Suppose f is continuous on the interval $[a, c]$ and on the interval $(c, b]$, where $a < c < b$, with a finite jump at $x = c$. Form a uniform partition on the interval $[a, c]$ with n grid points and another uniform partition on the interval $[c, b]$ with m grid points, where $x = c$ is a grid point of both partitions. Write a Riemann sum for $\int_a^b f(x)\,dx$ and separate it into two pieces for $[a, c]$ and $[c, b]$. Explain why $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

75–76. Piecewise continuous functions *Use geometry and the result of Exercise 74 to evaluate the following integrals.*

75. $\displaystyle\int_0^{10} f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{if } 0 \le x < 5 \\ 3 & \text{if } 5 \le x \le 10 \end{cases}$

76. $\displaystyle\int_1^{6} f(x)\,dx$ where $f(x) = \begin{cases} 2x & \text{if } 1 \le x < 4 \\ 10 - 2x & \text{if } 4 \le x \le 6 \end{cases}$

77. **Constants in integrals** Use the definition of the definite integral to justify the property $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where f is continuous and c is a real number.

78. **Exact area** Consider the linear function $f(x) = 2px + q$ on the interval $[a, b]$, where a, b, p, and q are positive constants.
    a. Show that the midpoint Riemann sum with n subintervals equals $\int_a^b f(x)\,dx$ for any value of n.
    b. Show that $\int_a^b f(x)\,dx = (b - a)[p(b + a) + q]$.

79. **A nonintegrable function** Consider the function defined on $[0, 1]$ such that $f(x) = 1$ if x is a rational number and $f(x) = 0$ if x is irrational. This function has an infinite number of discontinuities, and the integral $\int_0^1 f(x)\,dx$ does not exist. Show that if you consider only right, left, and midpoint Riemann sums on *regular* partitions with n subintervals, then they equal 1 for all n.

80. **Powers of x by Riemann sums** Consider the integral $I(p) = \int_0^1 x^p\,dx$ where p is a positive integer.
    a. Write the left Riemann sum for the integral with n subintervals.
    b. It is a fact (proved by the 17th-century mathematicians Fermat and Pascal) that $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p = \frac{1}{p+1}$. Use this fact to evaluate $I(p)$.

QUICK CHECK ANSWERS

1. -20 2. $f(x) = 1 - x$ is one possibility. 3. 0 4. 10; $c(b - a)$ 5. 0 6. $\frac{3}{2}; \frac{5}{2}$ ◄

5.3 Fundamental Theorem of Calculus

Evaluating definite integrals using limits of Riemann sums, as described in Section 5.2, is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals, which is developed in this section. Along the way, we discover the inverse relationship between differentiation and integration, expressed in the most important result of calculus, the Fundamental Theorem of Calculus.

Area Functions

The concept of an area function is crucial to the discussion about the connection between derivatives and integrals. We start with a continuous function $y = f(t)$ defined for $t \ge a$, where a is a fixed number. The *area function* for f with left endpoint a is denoted $A(x)$; it gives the net area of the region bounded by the graph of f and the t-axis between $t = a$ and $t = x$ (Figure 5.32). The net area of this region is also given by the definite integral

Independent variable of the area function

$$A(x) = \int_a^x f(t)\,dt.$$

Variable of integration (dummy variable)

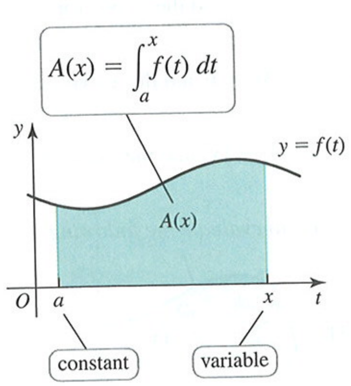


FIGURE 5.32

➤ A dummy variable is a placeholder; its role can be played by any symbol that does not conflict with other variables in the problem.

Notice that x is the upper limit of the integral *and* the independent variable of the area function: As x changes, so does the net area under the curve. Because the symbol x is already in use as the independent variable for A, we must choose another symbol for the variable of integration. Any symbol—except x—can be used because it is a *dummy variable*; we have chosen t as the integration variable.

Figure 5.33 gives a general view of how an area function is generated. Suppose that f is a continuous function and a is a fixed number. Now choose a point $b > a$. The net area of the region between the graph of f and the t-axis on the interval $[a, b]$ is $A(b)$. Moving the right endpoint to $(c, 0)$ or $(d, 0)$ produces different regions with net areas $A(c)$ and $A(d)$, respectively. In general, if $x > a$ is a variable point, then $A(x) = \int_a^x f(t)\,dt$ is the net area of the region between the graph of f and the t-axis on the interval $[a, x]$.

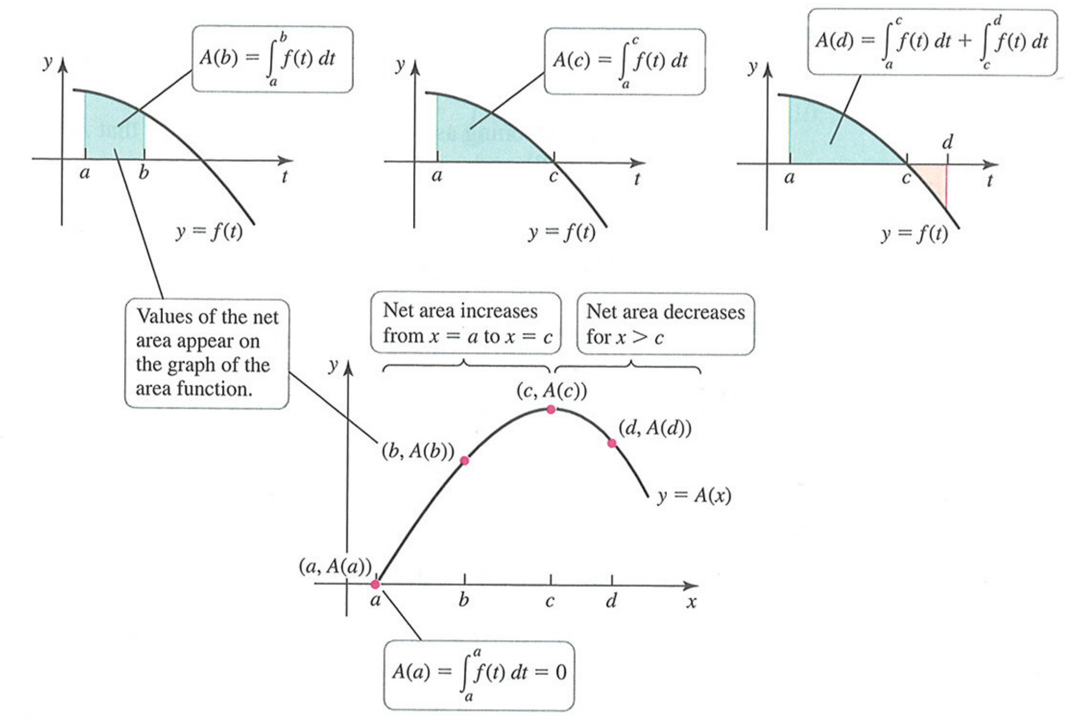


FIGURE 5.33

Figure 5.33 shows how $A(x)$ varies with respect to x. Notice that $A(a) = \int_a^a f(t)\,dt = 0$. Then, for $x > a$ the net area increases until $x = c$, at which point $f(c) = 0$. For $x > c$, the function f is negative, which produces a negative contribution to the area function. As a result, the area function decreases for $x > c$.

DEFINITION Area Function

Let f be a continuous function for $t \ge a$. The **area function for f with left endpoint a** is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \ge a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.